Quantum Hall Effect for Few Strongly Interacting Bosons at Finite Temperatures

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Abstract

Although the Chern number is a well-established topological invariant at zero temperature, its generalization to finite temperatures remains an active area of research. This work investigates the finite-temperature extension of the Chern number for few strongly interacting bosons. We demonstrate that the Chern number of the one-particle reduced density matrix is equivalent to the mixed Chern number, defined as the thermal average of Chern numbers over all eigenstates. This mixed Chern number corresponds to the visibility in quantum optics and is experimentally accessible. We propose three experimental schemes for Bose gases: two utilize time-reversal symmetry to directly extract monopole charges by measuring visibility while canceling dynamical phases, and one enables direct measurement of the mixed Chern number in the non-adiabatic regime. Our findings provide a practical framework for experimental validation of topological phenomena in finite-temperature quantum systems.

Monopoles and Mixed Chern Numbers

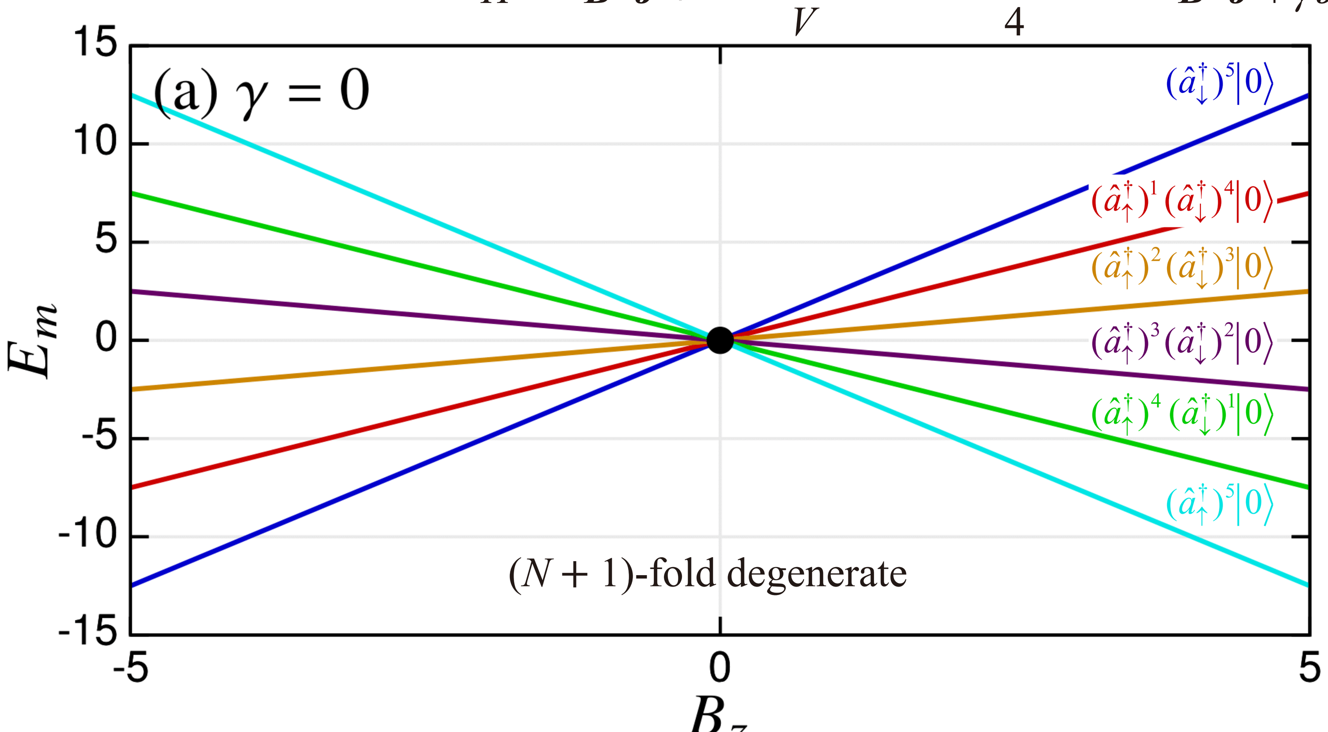
We consider N bosons in two hyperfine states coupled with lasers (or N bosons in two lattice sites) with strong inter hyperfine interaction (or strong on-site interaction).

By the spatial single-mode approximation, the Hamiltonian reads

$$\hat{H} = E_\uparrow \hat{a}_\uparrow^\dagger \hat{a}_\uparrow + E_\downarrow \hat{a}_\downarrow^\dagger \hat{a}_\downarrow + \frac{\Omega}{2}\hat{a}_\uparrow^\dagger \hat{a}_\downarrow + \frac{\Omega^*}{2}\hat{a}_\downarrow^\dagger \hat{a}_\uparrow + \frac{g_{\text{intra}}}{V}(\hat{N}_\uparrow^2 + \hat{N}_\downarrow^2) + \frac{g_{\text{inter}}}{V}\hat{N}_\uparrow \hat{N}_\downarrow.$$

After Jordan–Schwinger map [1, 2], the whole system is mapped onto a system of spin-$N/2$ particles subjected to an effective magnetic field with effective interacting Hamiltonian

$$\hat{H} = -\boldsymbol{B}\cdot\hat{\boldsymbol{J}} + \frac{2g_{\text{intra}} - g_{\text{inter}}}{V}\frac{(\hat{N}_\uparrow^2 - \hat{N}_\downarrow^2)^2}{4} = -\boldsymbol{B}\cdot\hat{\boldsymbol{J}} + \gamma \hat{J}_z^2.$$



Along the z-direction, the spin states $(\hat{a}_\uparrow^\dagger)^{N_\uparrow}(\hat{a}_\downarrow^\dagger)^{N-N_\uparrow}|0\rangle = |N/2, m\rangle$, where $m = (2N_\uparrow - N)/2$, are the eigenstates with

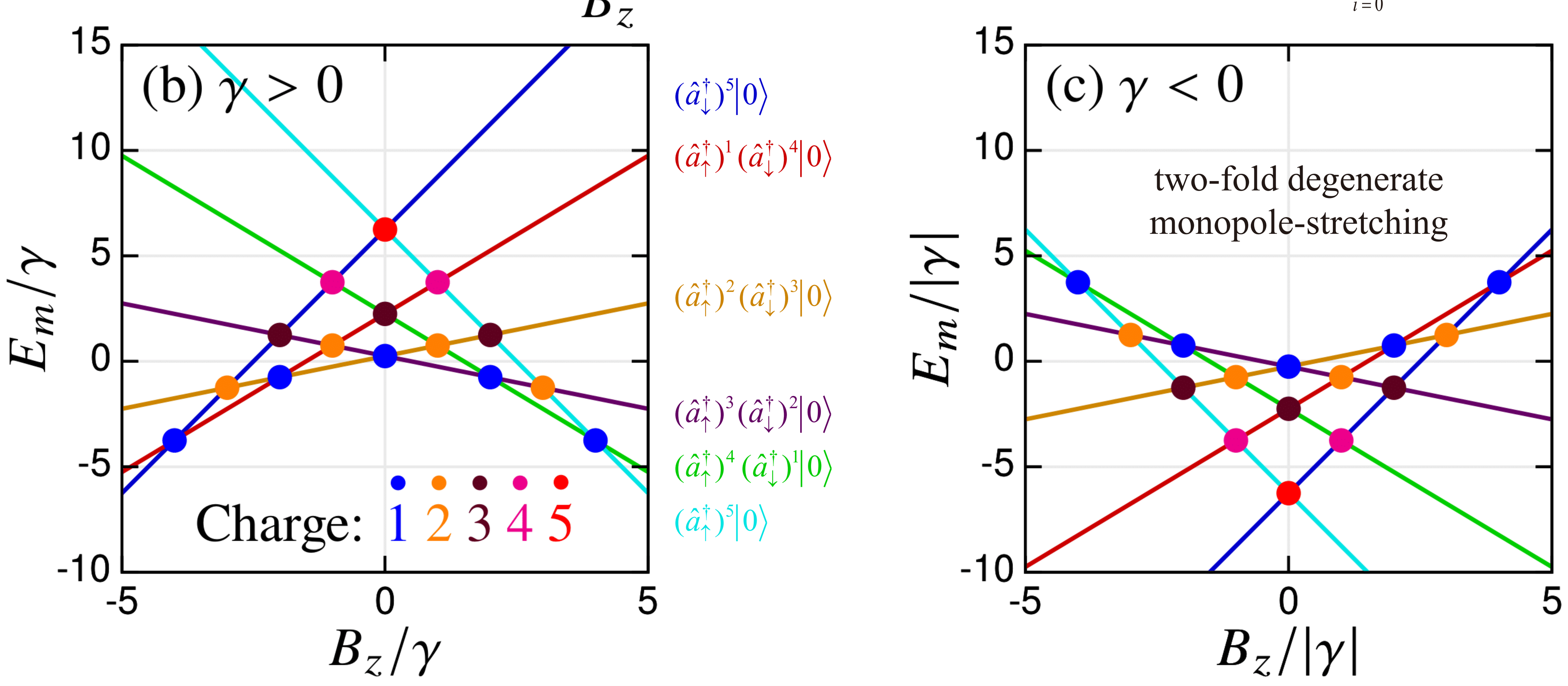
$$E_m = -B_z m + \gamma m^2.$$

The Chern number of the i-th state is

$$C_i = \text{sgn}(\gamma)\,(N - 2i).$$

To account for the finite-temperature effect, we define the mixed Chern number as

$$C_{\text{mixed}} = \sum_{i=0}^{N} p_i C_i.$$



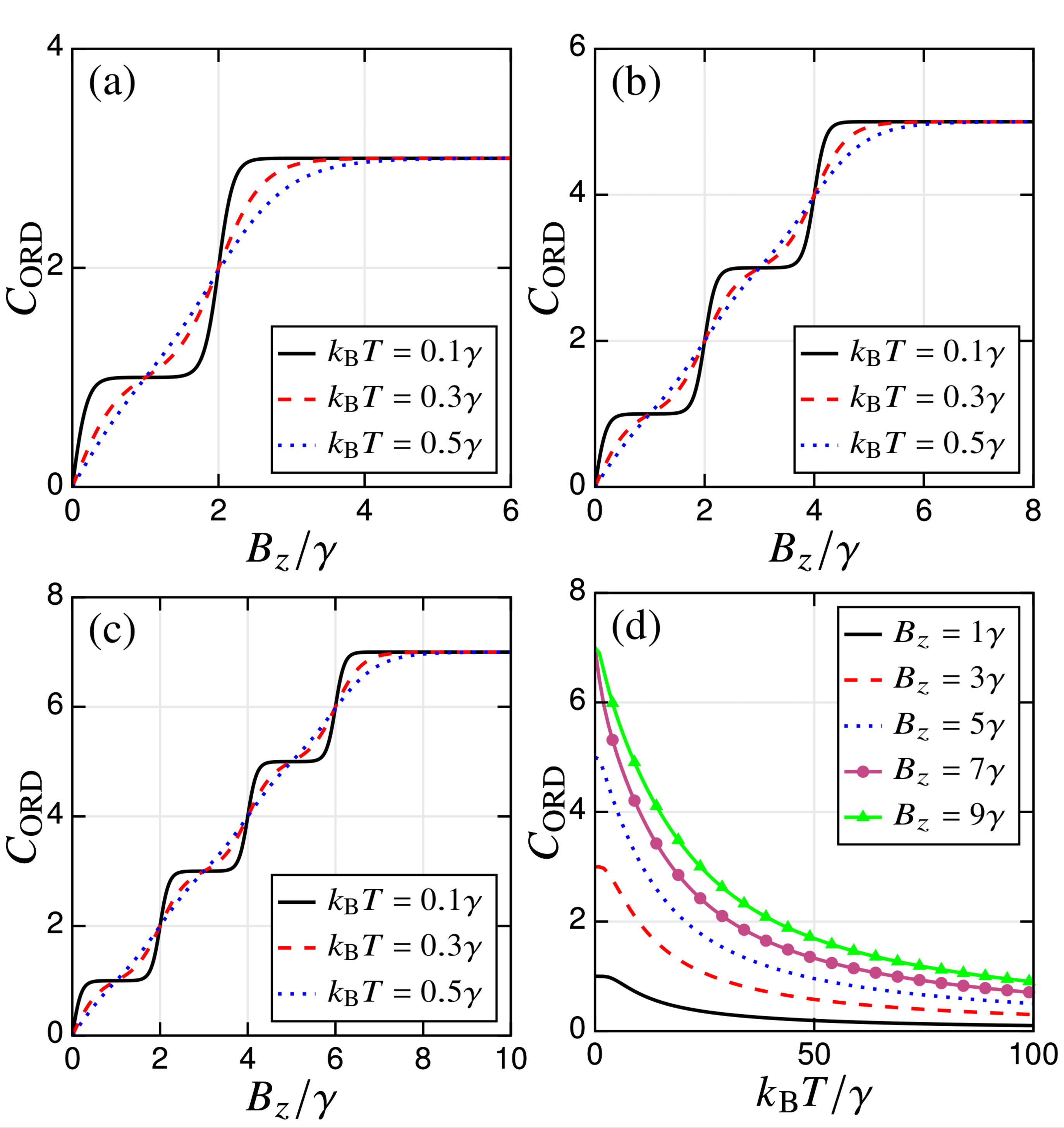
Measurements of the Mixed Chern Numbers

Sjöqvist et al. defined the mixed geometric phase [3]. For a spin-1/2 system, it is defined by

$$v e^{i\gamma_{\text{mixed}}} = p_1 e^{-i\Omega} + p_2 e^{i\Omega}.$$

v: visibility γ_{mixed}: mixed geometric phase Ω: solid angle subtended

For a thermal state and $\Omega = \pi$, the visibility is $v = \tanh(B/2k_BT)$, which coincides with C_{mixed}.



For $N > 1$ particles, the mixed Chern numbers and visibility are related by the one-particle reduced density matrix (ORD)

$$\rho^{(1)} = \begin{bmatrix} \langle \hat{a}_\uparrow^\dagger \hat{a}_\uparrow \rangle & \langle \hat{a}_\uparrow^\dagger \hat{a}_\downarrow \rangle \\ \langle \hat{a}_\downarrow^\dagger \hat{a}_\uparrow \rangle & \langle \hat{a}_\downarrow^\dagger \hat{a}_\downarrow \rangle \end{bmatrix}.$$

New definition:

$$C_{\text{ORD}} = \text{tr}(\rho^{(1)}\sigma_z).$$

Two definitions of Chern numbers are equivalent,

$$C_{\text{ORD}} = C_{\text{mixed}}.$$

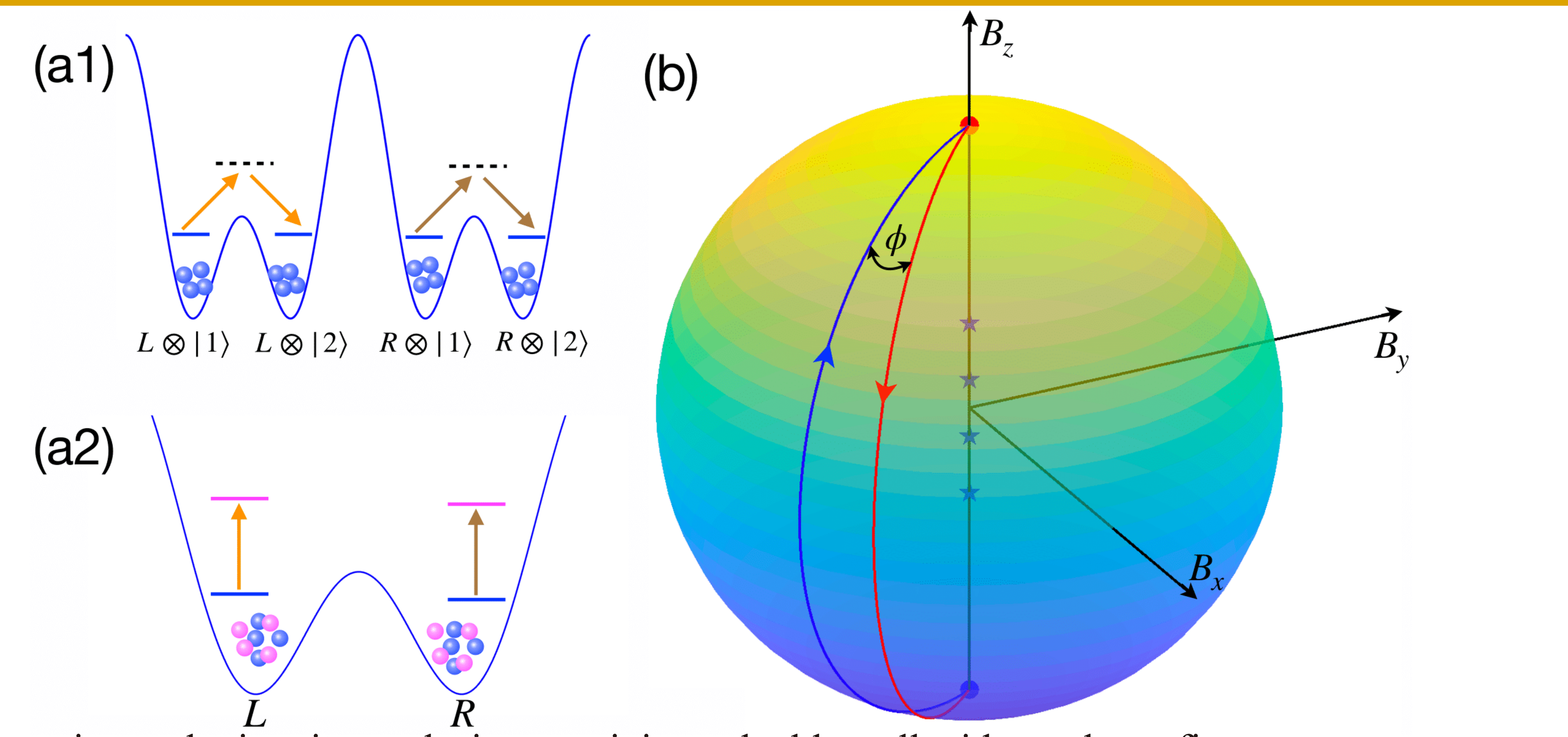
The mixed geometric phase for the one-particle density matrix is defined by

$$v_{\text{ORD}} e^{i\gamma_{\text{ORD}}} = \rho_{1,1}^{(1)} e^{-i\Omega} + \rho_{2,2}^{(1)} e^{i\Omega}.$$

When $\Omega = \pi$, we have

$$v_{\text{ORD}} = \rho_{1,1}^{(1)} - \rho_{2,2}^{(1)} = C_{\text{ORD}}.$$

Experimental Proposals



(a1) Bose gas in two lattice site, each site containing a double well with two hyperfine states.

(a2) Two-component Bose gas in a double well with two hyperfine states.

The effective magnetic fields in each site or well are oriented in opposite direction. The effective Hamiltonian reads

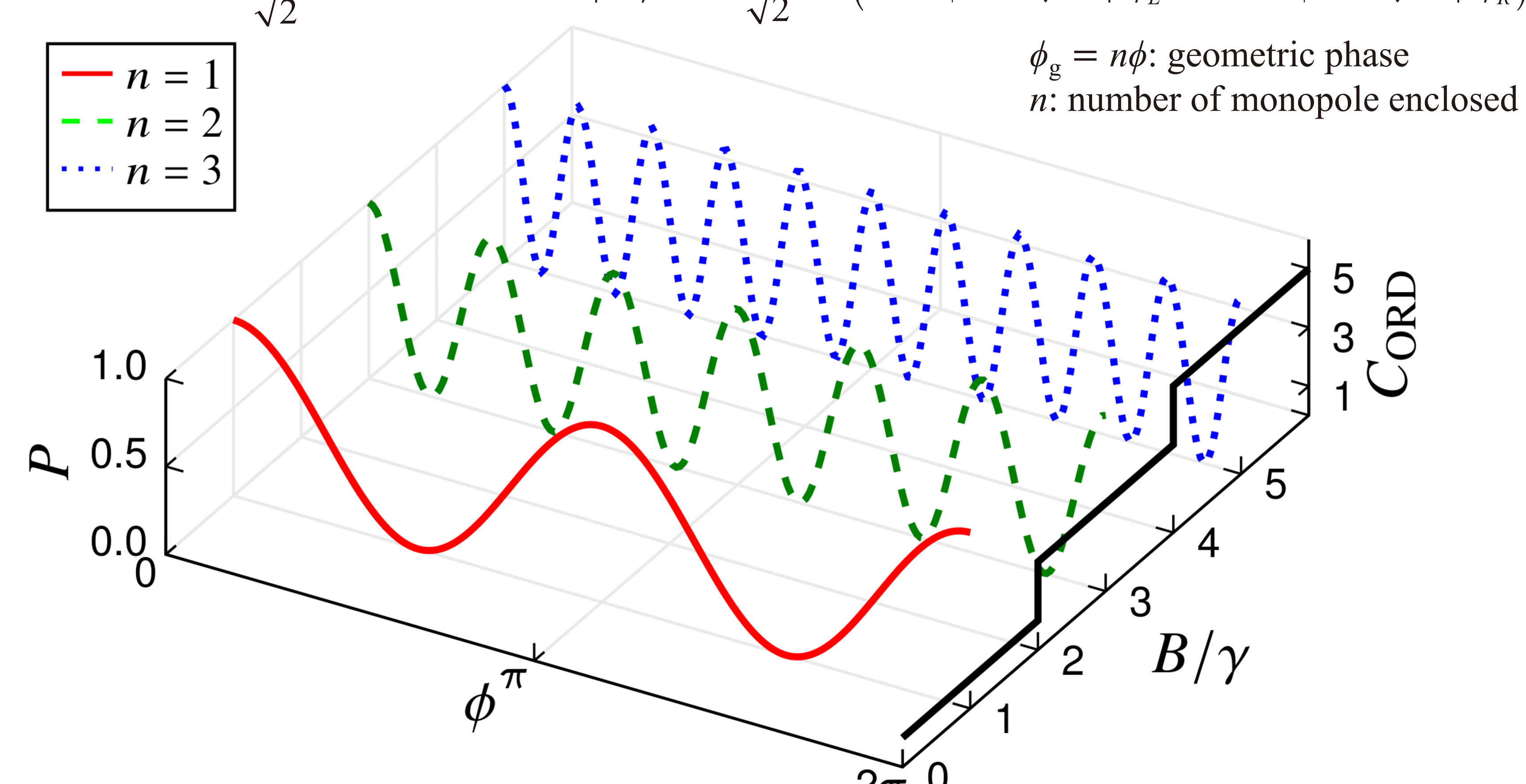
$$\hat{H} = \begin{bmatrix} -\boldsymbol{B}\cdot\hat{\boldsymbol{J}} + \gamma\hat{J}_z^2 & \hat{0} \\ \hat{0} & \boldsymbol{B}\cdot\hat{\boldsymbol{J}} + \gamma\hat{J}_z^2 \end{bmatrix},$$

which is time-reversal symmetric, so that the dynamical phase can be canceled. We evolve the ground state along the evolution path as shown in above figure. We have

$$|\Psi_i\rangle = \frac{(\hat{a}_\uparrow^\dagger)^{N_\uparrow}(\hat{a}_\downarrow^\dagger)^{N_\downarrow}|0\rangle_L \oplus (\hat{a}_\uparrow^\dagger)^{N_\downarrow}(\hat{a}_\downarrow^\dagger)^{N_\uparrow}|0\rangle_R}{\sqrt{2}} \to |\Psi_f\rangle = \frac{e^{-i\int_0^{t_f} E_m(t')dt'}}{\sqrt{2}}\left(e^{i\phi_g}(\hat{a}_\uparrow^\dagger)^{N_\uparrow}(\hat{a}_\downarrow^\dagger)^{N\downarrow}|0\rangle_L \oplus e^{-i\phi_g}(\hat{a}_\uparrow^\dagger)^{N_\downarrow}(\hat{a}_\downarrow^\dagger)^{N_\uparrow}|0\rangle_R\right).$$

$\phi_g = n\phi$: geometric phase

n: number of monopole enclosed



After the time of flight, we measure

$$P = |\langle \Psi_f | \Psi_i \rangle|^2 = \cos^2(n\phi).$$

By varying the angle ϕ, we can extract the monopole charge n.

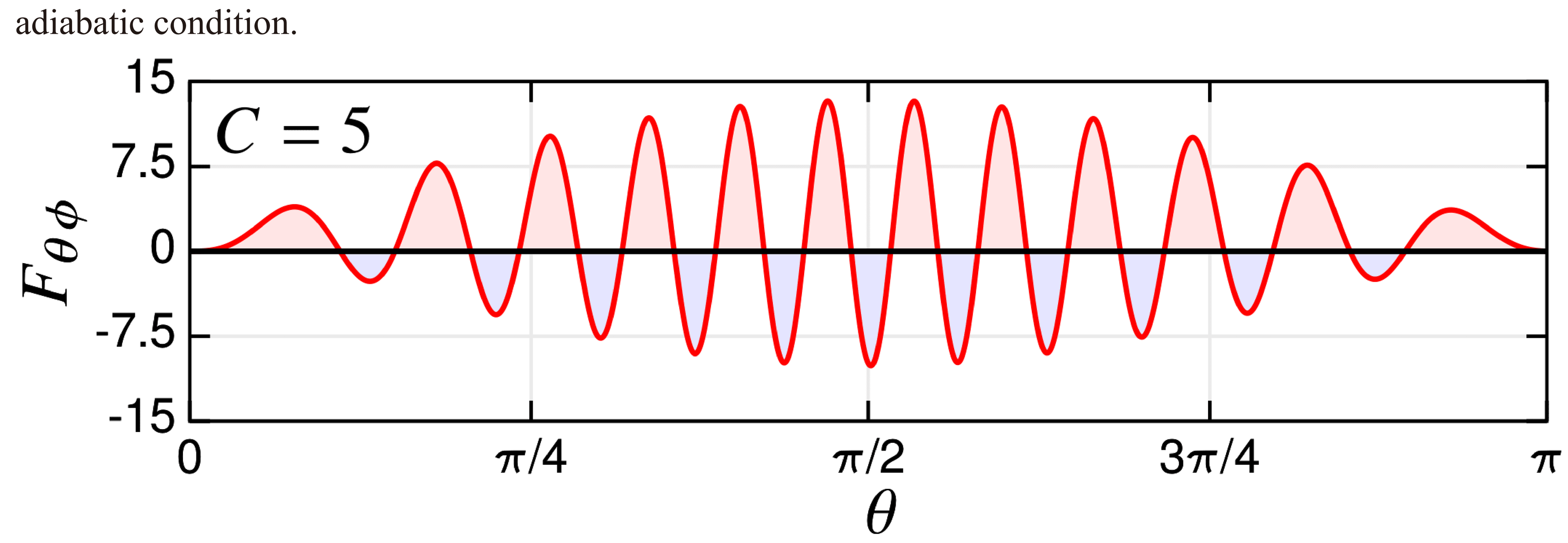
We can also measure the Chern number of the ground state by integrating the linear response in the non-adiabatic limit [4]. We evolve the ground state along a meridian on the sphere with $\theta(t) = \pi t/t_{\text{total}}$, $\phi(t) = 0$. At each time step, we measure the Berry curvature

$$F_{\theta\phi} = \frac{B\sin(\theta)t_{\text{total}}}{\pi}\langle J_y \rangle$$

via tomography [5]. The Chern number can then be obtained by integrating the Berry curvature,

$$C = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi} F_{\theta\phi}\,d\theta\,d\phi = \int_0^{\pi} F_{\theta\phi}\,d\theta.$$

Due to finite-temperature effects, we are measuring the ensemble average of $\langle J_y \rangle$ for a thermal state, so the measured value corresponds to the mixed Chern number C_{mixed}. This method does not require a strict adiabatic condition.



This quantization can be interpreted as the quantum Hall effect in the abstract parameter space for interacting bosons. By conceptualizing the quench velocity v_θ as electric field, the generalized force $\langle f_\phi \rangle$ as electric current, and the Berry curvature $F_{\theta\phi}$ as Hall conductivity, we can draw parallels to the behavior observed in the quantum Hall effect [4].

It is important to note that the Berry curvature is directly related to $\langle J_y \rangle$, which describes the exchange or flow of particles between the two modes of the Bose gas. When averaged over the sphere, the Chern number effectively quantifies the difference in occupation numbers between the two modes, analogous to the quantized Hall conductance in the quantum Hall effect.

References and Acknowledgements

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